An introduction to INLA with a comparison to JAGS

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Bayes 2013
Rotterdam, Tuesday 21 May 2013
Laplace’s liberation army (?)

Introduction to INLA

Bayes 2013, 21 May 2013
Outline of presentation

1 9.00 – 9.45: (Quick & moderately clean) introduction to Bayesian computation
   - MCMC
   - Latent Gaussian models
   - Gaussian Markov Random Fields

2 9.45 – 10.00: Coffee break

3 10.00 – 10.45: Introduction to INLA
   - Basic ideas
   - Some details
   - A simple example

4 10.45 – 11.00: Coffee break

5 11.00 – 12.00: Using the package R-INLA
   - How does it work?
   - Some simple examples
   - (Slightly) more complex examples
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(Quick & moderately clean) introduction to Bayesian computation
Bayesian computation

- In a *(very small!)* nutshell, Bayesian inference boils down to the computation of posterior and/or predictive distributions

\[
p(\theta \mid y) = \frac{p(y \mid \theta)p(\theta)}{\int p(y \mid \theta)p(\theta)d\theta}, \quad p(y^* \mid y) = \int p(y^* \mid \theta)p(\theta \mid y)d\theta
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Since the advent of simulation-based techniques (notably MCMC), Bayesian computation has enjoyed incredible development.

This has certainly been helped by dedicated software (eg BUGS and then WinBUGS, OpenBUGS, JAGS).

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• MCMC methods are very general and can effectively be applied to “any” model

• However:
  – Even if in theory, MCMC can provide (nearly) exact inference, given perfect convergence and MC error \( \rightarrow 0 \), in practice, this has to be balanced with model complexity and running time
  – This is particularly an issue for problems characterised by large data or very complex structure (eg hierarchical models)
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  1. Select a set of initial values \( (\theta_1^{(0)}, \theta_2^{(0)}, \ldots, \theta_J^{(0)}) \)

  2. Sample \( \theta_1^{(1)} \) from the conditional distribution \( p(\theta_1 | \theta_2^{(0)}, \theta_3^{(0)}, \ldots, \theta_J^{(0)}, y) \)

     Sample \( \theta_2^{(1)} \) from the conditional distribution \( p(\theta_2 | \theta_1^{(1)}, \theta_3^{(0)}, \ldots, \theta_J^{(0)}, y) \)

     \( \ldots \)

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   \[ \vdots \]
   Sample $\theta_J^{(1)}$ from the conditional distribution $p(\theta_J | \theta_1^{(1)}, \theta_2^{(1)}, \ldots, \theta_{J-1}^{(1)}, y)$.
3. Repeat step 2. for $S$ times until convergence is reached to the target distribution $p(\theta | y)$.
4. Use the sample from the target distribution to compute all relevant statistics: (posterior) mean, variance, credibility intervals, etc.

If the full conditionals are not readily available, they need to be estimated (e.g., via Metropolis-Hastings or slice sampling) before applying the GS.
After 10 iterations
MCMC — convergence

After 30 iterations

\begin{figure}
\centering
\includegraphics[width=\textwidth]{mcmc_convergence.png}
\end{figure}
After 1000 iterations
MCMC — convergence

![Graph showing convergence of MCMC chains with burn-in and sample after convergence](image)

- **Iteration**
- **Burn-in**
- **Sample after convergence**
Formal assessment of convergence: potential scale reduction

\[ \hat{R} = \sqrt{\frac{\widehat{\text{Var}}(\theta_k | y)}{W(\theta_k)}} \]
- Formal assessment of autocorrelation: effective sample size

\[ n_{\text{eff}} = \frac{S}{1 + 2 \sum_{t=1}^{\infty} \rho_t} \]
MCMC — brute force

Uncentred model with thinning

Autocorrelation function for $\alpha$ – Uncentred model with thinning

$\alpha$

$\beta$

Iteration

Lag
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   - No U-turn sampling (e.g. stan — more on this later!)
MCMC — pros & cons

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  3 Alternative methods of inference
     • Approximate Bayesian Computation (ABC)
     • INLA
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   - INLA
The basic ideas revolve around

- Formulating the model using a specific characterisation
  - All models that can be formulated in this way have certain features in common, which facilitate the computational aspects
  - The characterisation is still quite general and covers a wide range of possible models (more on that later!)
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- Compute the relevant quantities typically using numerical methods
Latent Gaussian models (LGMs)

- The general problem of (parametric) inference is posited by assuming a probability model for the observed data, as a function of some relevant parameters

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- Often (in fact for a surprisingly large range of models!), we can assume that the parameters are described by a **Gaussian Markov Random Field (GMRF)**

\[ \theta \mid \psi \sim \text{Normal}(0, \Sigma(\psi)) \]

\[ \theta_l \perp \perp \theta_m \mid \theta_{-lm} \]

where

- The notation “\(-lm\)” indicates all the other elements of the parameters vector, excluding elements \( l \) and \( m \)
- The covariance matrix \( \Sigma \) depends on some hyper-parameters \( \psi \)
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- This kind of models is often referred to as **Latent Gaussian models**
In general, we can partition $\psi = (\psi_1, \psi_2)$ and re-express a LGM as

$$\psi \sim p(\psi) \quad \text{("hyperprior")}$$

$$\theta | \psi \sim p(\theta | \psi) = \text{Normal}(0, \Sigma(\psi_1)) \quad \text{("GMRF prior")}$$

$$y | \theta, \psi \sim \prod_i p(y_i | \theta, \psi_2) \quad \text{("data model")}$$

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LGMs as a general framework

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ie \( \psi_1 \) are the **hyper-parameters** and \( \psi_2 \) are **nuisance parameters**

- The dimension of \( \theta \) can be very large (eg \( 10^2-10^5 \))
- Conversely, because of the conditional independence properties, the dimension of \( \psi \) is generally small (eg 1-5)
LGMs as a general framework

- A very general way of specifying the problem is by modelling the mean for the $i$-th unit by means of an additive linear predictor, defined on a suitable scale (e.g. logistic for binomial data)

$$\eta_i = \alpha + \sum_{m=1}^{M} \beta_m x_{mi} + \sum_{l=1}^{L} f_l(z_{li})$$

where
- $\alpha$ is the intercept;
- $\beta = (\beta_1, \ldots, \beta_M)$ quantify the effect of $x = (x_1, \ldots, x_M)$ on the response;
- $f = \{f_1(\cdot), \ldots, f_L(\cdot)\}$ is a set of functions defined in terms of some covariates $z = (z_1, \ldots, z_L)$

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**NB**: This of course implies some form of Normally-distributed marginals for $\alpha$, $\beta$ and $f$. 
LGMs as a general framework — examples

Upon varying the form of the functions $f_1(\cdot)$, this formulation can accommodate a wide range of models.
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  - $\sigma_f^2 | \psi \sim \text{some common distribution}$
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- Spatial and spatio-temporal models
  - Two components: \( f_1(\cdot) \sim \text{CAR} \) (Spatially structured effects)
    \[ f_2(\cdot) \sim \text{Normal}(0, \sigma_{f_2}^2) \] (Unstructured residual)
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- **Spline smoothing**
  - $f_l(\cdot) \sim \text{AR}(\phi, \sigma_\varepsilon^2)$
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- **Survival models / logGaussian Cox Processes**
  - More complex specification in theory, but relatively easy to fit within the INLA framework!
Gaussian Markov Random Field

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- Generally, it is complicated to do it in terms of the covariance matrix $\Sigma$
  - Typically, $\Sigma$ is dense (ie many of the entries are non-zero)
  - If it happens to be sparse, this implies (marginal) independence among the relevant elements of $\theta$ — this is generally too stringent a requirement!
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- Conversely, it is much simpler when using the precision matrix $Q := \Sigma^{-1}$
  - As it turns out, it can be shown that
    \[ \theta_i \perp \perp \theta_m \mid \theta_{-lm} \iff Q_{lm} = 0 \]
  - Thus, under conditional independence (which is a less restrictive constraint), the precision matrix is typically sparse
  - We can use existing numerical methods to deal with sparse matrices (e.g., the R package Matrix)
  - Most computations in GMRFs are performed in terms of computing Cholesky’s factorisations
Precision matrix and conditional independence
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• (Standard) MCMC methods tend to perform poorly when applied to (non-trivial) LGMs. This is due to several factors
  – The components of the latent Gaussian field $\theta$ tend to be highly correlated, thus impacting on convergence and autocorrelation
  – Especially when the number of observations is large, $\theta$ and $\psi$ also tend to be highly correlated

\[ \rho = 0.95 \]
\[ \rho = 0.20 \]
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- The components of the latent Gaussian field $\theta$ tend to be highly correlated, thus impacting on convergence and autocorrelation.
- Especially when the number of observations is large, $\theta$ and $\psi$ also tend to be highly correlated.

Again, blocking and overparameterisation can *alleviate*, but rarely eliminate the problem.
**Summary so far**

- Bayesian computation (especially for LGMs) is in general difficult
- MCMC can be efficiently used in many simple cases, but becomes a bit trickier for complex models
  - Issues with convergence
  - Time to run can be very long
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- A wide class of statistical models can be represented in terms of LGM
- This allows us to take advantage of nice computational properties
  - GMRFs
  - Sparse precision matrices

- This is at the heart of the INLA approach
Introduction to INLA
The starting point to understand the INLA approach is the definition of conditional probability, which holds for any pair of variables \((x, z)\) — and, technically, provided \(p(z) > 0\)

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In particular, a conditional version can be obtained further considering a third variable \(w\) as

\[
p(z \mid w) = \frac{p(x, z \mid w)}{p(x \mid z, w)}
\]

which is particularly relevant to the Bayesian case
The second “ingredient” is **Laplace approximation**
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Main idea: approximate $\log g(x)$ using a quadratic function by means of a Taylor’s series expansion around the mode $\hat{x}$

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\log g(x) \approx \log g(\hat{x}) + \frac{\partial \log g(\hat{x})}{\partial x} (x - \hat{x}) + \frac{1}{2} \frac{\partial^2 \log g(\hat{x})}{\partial x^2} (x - \hat{x})^2
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= \log g(\hat{x}) + \frac{1}{2} \frac{\partial^2 \log g(\hat{x})}{\partial x^2} (x - \hat{x})^2 \quad \left(\text{since } \frac{\partial \log g(\hat{x})}{\partial x} = 0\right)
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Setting $\hat{\sigma}^2 = -\frac{1}{\partial^2 \log g(\hat{x})/\partial x^2}$ we can re-write

$$\log g(x) \approx \log g(\hat{x}) - \frac{1}{2\hat{\sigma}^2}(x - \hat{x})^2$$

or equivalently

$$\int g(x) dx = \int e^{\log g(x)} dx \approx \text{const} \int \exp \left[-\frac{(x - \hat{x})^2}{2\hat{\sigma}^2}\right] dx$$
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Thus, under LA, $g(x) \approx \text{Normal}(\hat{x}, \hat{\sigma}^2)$
Consider a $\chi^2$ distribution: $p(x) = \frac{g(x)}{c} = \frac{x^{\frac{k}{2}-1} e^{-\frac{x}{2}}}{c}$
Laplace approximation — example

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1. $l(x) = \log g(x) = \left(\frac{k}{2} - 1\right) \log x - \frac{x}{2}$
2. $l'(x) = \frac{\partial \log g(x)}{\partial x} = \left(\frac{k}{2} - 1\right) x^{-1} - \frac{1}{2}$
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  \[ p(x) = \frac{g(x)}{c} = \frac{x^{k/2-1}e^{-x/2}}{c} \]

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2. \[ l'(x) = \frac{\partial \log g(x)}{\partial x} = \left( \frac{k}{2} - 1 \right) x^{-1} - \frac{1}{2} \]
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Laplace approximation — example

- Consider a $\chi^2$ distribution: 
  $$ p(x) = \frac{g(x)}{c} = \frac{x^{k-1}e^{-\frac{x}{2}}}{c} $$

1. $l(x) = \log g(x) = \left(\frac{k}{2} - 1\right) \log x - \frac{x}{2}$
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- Then
  - Solving $l'(x) = 0$ we find the mode: $\hat{x} = k - 2$
  - Evaluating $\frac{1}{l''(x)}$ at the mode gives $\hat{\sigma}^2 = 2(k - 2)$
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Then

- Solving $l'(x) = 0$ we find the mode: $\hat{x} = k - 2$
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Consequently, we can approximate $p(x)$ as

$$p(x) \approx \tilde{p}(x) = \text{Normal}(k - 2, 2(k - 2))$$
Laplace approximation — example

- $\chi^2(3)$
- Normal(1, 2)

- $\chi^2(6)$
- Normal(4, 8)

- $\chi^2(10)$
- Normal(8, 16)

- $\chi^2(20)$
- Normal(18, 36)
• The general idea is that using the fundamental probability equations, we can approximate a generic conditional (posterior) distribution as

\[
\tilde{p}(z \mid w) = \frac{p(x, z \mid w)}{\tilde{p}(x \mid z, w)},
\]

where \(\tilde{p}(x \mid z, w)\) is the Laplace approximation to the conditional distribution of \(x\) given \(z, w\).
The general idea is that using the fundamental probability equations, we can approximate a generic conditional (posterior) distribution as

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This idea can be used to approximate any generic required posterior distribution.
Objective of Bayesian estimation

- In a Bayesian LGM, the required distributions are

\[ p(\theta_j \mid y) = \int p(\theta_j, \psi \mid y) d\psi = \int p(\psi \mid y)p(\theta_j \mid \psi, y) d\psi \]

\[ p(\psi_k \mid y) = \int p(\psi \mid y) d\psi_{-k} \]
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  (1.) \( p(\psi \mid y) \), from which also all the relevant marginals \( p(\psi_k \mid y) \) can be obtained;
Objective of Bayesian estimation

- In a Bayesian LGM, the required distributions are

\[
p(\theta_j | y) = \int p(\theta_j, \psi | y) d\psi = \int p(\psi | y) p(\theta_j | \psi, y) d\psi
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\[
p(\psi_k | y) = \int p(\psi | y) d\psi_{-k}
\]

- Thus we need to estimate:
  
  (1.) \( p(\psi | y) \), from which also all the relevant marginals \( p(\psi_k | y) \) can be obtained;

  (2.) \( p(\theta_j | \psi, y) \), which is needed to compute the marginal posterior for the parameters
(1.) can be easily estimated as

\[ p(\psi \mid y) = \frac{p(\theta, \psi \mid y)}{p(\theta \mid \psi, y)} \]
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\[
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\[ \propto \frac{p(\psi)p(\theta | \psi)p(y | \theta)}{p(\theta | \psi, y)} \]

\[ \approx \frac{p(\psi)p(\theta | \psi)p(y | \theta)}{\tilde{p}(\theta | \psi, y)} \bigg|_{\theta = \hat{\theta}(\psi)} =: \tilde{p}(\psi | y) \]

where

- \( \tilde{p}(\theta | \psi, y) \) is the Laplace approximation of \( p(\theta | \psi, y) \)
- \( \theta = \hat{\theta}(\psi) \) is its mode
(2.) is slightly more complex, because in general there will be more elements in \( \theta \) than there are in \( \psi \) and thus this computation is more expensive.
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\[\propto \frac{p(\theta, \psi | y)}{p(\theta_{-j} | \theta_j, \psi, y)} = \frac{p(\psi)p(\theta | \psi)p(y | \theta)}{p(\theta_{-j} | \theta_j, \psi, y)}\]
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$$p(\theta_j \mid \psi, y) \propto \frac{p(\theta, \psi \mid y)}{p(\theta_{-j} \mid \theta_j, \psi, y)} \propto \frac{p(\psi)p(\theta \mid \psi)p(y \mid \theta)}{p(\theta_{-j} \mid \theta_j, \psi, y)}$$

$$\approx \frac{p(\psi)p(\theta \mid \psi)p(y \mid \theta)}{\tilde{p}(\theta_{-j} \mid \theta_j, \psi, y)} \bigg|_{\theta_{-j}=\hat{\theta}_{-j}(\theta_j, \psi)} =: \tilde{p}(\theta_j \mid \psi, y)$$
• Because \((\theta_{-j} \mid \theta_j, \psi, y)\) are reasonably Normal, the approximation works generally well
• However, this strategy can be computationally expensive
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• The most efficient algorithm is the “Simplified Laplace Approximation”
  – Based on a Taylor’s series expansion up to the third order of both numerator and denominator for $\tilde{p}(\theta_j \mid \psi, y)$
  – This effectively “corrects” the Gaussian approximation for location and skewness to increase the fit to the required distribution
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• This is the algorithm implemented by default by R-INLA, but this choice can be modified
  – If extra precision is required, it is possible to run the full Laplace approximation — of course at the expense of running time!
Operationally, the INLA algorithm proceeds with the following steps:

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   - Compute the Hessian at $\hat{\psi}$ and change co-ordinates to standardise the variables; this corrects for scale and rotation and simplifies integration

\[
\begin{align*}
\mathbb{E}[z] &= 0 \\
V[z] &= \sigma^2 I
\end{align*}
\]
Integrated Nested Laplace Approximation (INLA)

Operationally, the INLA algorithm proceeds with the following steps:

i. Explore the marginal joint posterior for the hyper-parameters \( \tilde{p}(\psi \mid y) \)
   - Locate the mode \( \hat{\psi} \) by optimising \( \log \tilde{p}(\psi \mid y) \), eg using Newton-like algorithms
   - Compute the Hessian at \( \hat{\psi} \) and change co-ordinates to standardise the variables; this corrects for scale and rotation and simplifies integration
   - Explore \( \log \tilde{p}(\psi \mid y) \) and produce a grid of \( H \) points \( \{\psi^*_h\} \) associated with the bulk of the mass, together with a corresponding set of area weights \( \{\Delta_h\} \)
ii. For each element $\psi_h^*$ in the grid,

- Obtain the marginal posterior $\tilde{p}(\psi_h^* \mid y)$, using interpolation and possibly correcting for (probable) skewness by using log-splines;
- Evaluate the conditional posteriors $\tilde{p}(\theta_j \mid \psi_h^*, y)$ on a grid of selected values for $\theta_j$;
ii. For each element $\psi^*_h$ in the grid,
   - Obtain the marginal posterior $\tilde{p}(\psi^*_h \mid y)$, using interpolation and possibly correcting for (probable) skewness by using log-splines;
   - Evaluate the conditional posteriors $\tilde{p}(\theta_j \mid \psi^*_h, y)$ on a grid of selected values for $\theta_j$;

iii. Marginalise $\psi^*_h$ to obtain the marginal posteriors $\tilde{p}(\theta_j \mid y)$ using numerical integration

$$\tilde{p}(\theta_j \mid y) \approx \sum_{h=1}^{H} \tilde{p}(\theta_j \mid \psi^*_h, y)\tilde{p}(\psi^*_h \mid y)\Delta_h$$
So, it’s all in the name…

Integrated Nested Laplace Approximation

- Because Laplace approximation is the basis to estimate the unknown distributions
- Because the Laplace approximations are nested within one another
  - Since (2.) is needed to estimate (1.)
  - NB: Consequently the estimation of (1.) might not be good enough, but it can be refined
- Because the required marginal posterior distributions are obtained by (numerical) integration
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Suppose we want to make inference on a very simple model

\[ y_{ij} \mid \theta_j, \psi \sim \text{Normal}(\theta_j, \sigma_0^2) \quad (\sigma_0^2 \text{ assumed known}) \]
\[ \theta_j \mid \psi \sim \text{Normal}(0, \tau) \quad (\psi = \tau^{-1} \text{ is the precision}) \]
\[ \psi \sim \text{Gamma}(a, b) \]
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\end{align*}
\]

So, the model is made by a three-level hierarchy:

1. Data \( y = (y_{ij}) \) for \( i = 1, \ldots, n_j \) and \( j = 1, \ldots, J \)
2. Parameters \( \theta = (\theta_1, \ldots, \theta_J) \)
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So, the model is made by a three-level hierarchy:

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2. Parameters \( \theta = (\theta_1, \ldots, \theta_J) \)
3. Hyper-parameter \( \psi \)

**NB**: This model is in fact semi-conjugated, so inference is possible numerically or using simple MCMC algorithms
Because of semi-conjugacy, we know that

\[ \theta, y \mid \psi \sim \text{Normal}(\cdot, \cdot) \]

and thus we can compute (numerically) all the marginals.
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In particular

\[
p(\psi \mid y) \propto p(y \mid \psi)p(\psi)
\]

\[
\propto \frac{p(\theta, y \mid \psi)p(\psi)}{p(\theta \mid y, \psi)}
\]

\[
\text{Gaussian}
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Because of semi-conjugacy, we know that

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and thus we can compute (numerically) all the marginals.

In particular

\[
p(\psi \mid y) \propto p(y \mid \psi)p(\psi)
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Moreover, because \( p(\theta \mid y) \sim \text{Normal}(\cdot, \cdot) \) and so are all the resulting marginals (ie for every element \( j \)), it is easy to compute

\[
p(\theta_j \mid y) = \int p(\theta_j \mid y, \psi) \ p(\psi \mid y) \ d\psi
\]
1. Select a grid of $H$ points for $\psi (\{\psi^*_h\})$ and the associated area weights ($\{\Delta_h\}$)

Posterior marginal for $\psi : p(\psi \mid y) \propto \frac{p(\theta, y \mid \psi)p(\psi)}{p(\theta \mid y, \psi)}$
2. Interpolate the posterior density to compute the approximation to the posterior
3. Compute the posterior marginal for each $\theta_j$ given each $\psi$ on the $H$-dimensional grid
4. Weight the resulting (conditional) marginal posteriors by the density associated with each \( \psi \) on the grid.

Posterior marginal for \( \theta_1 \), conditional on each \( \{\psi_h^*\} \) value (weighted)
5. (Numerically) sum over all the conditional densities to obtain the marginal posterior for each of the elements $\theta_j$
The basic idea behind the INLA procedure is simple

- Repeatedly use Laplace approximation and take advantage of computational simplifications due to the structure of the model
- Use numerical integration to compute the required posterior marginal distributions
- (If necessary) refine the estimation (eg using a finer grid)
The basic idea behind the INLA procedure is simple
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- (If necessary) refine the estimation (e.g., using a finer grid)

Complications are mostly computational and occur when
- Extending to more than one hyper-parameter
- Markedly non-Gaussian observations
Using the package \texttt{R-INLA}
Good news is that all the procedures needed to perform INLA are implemented in a R package. This is effectively made by two components.
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1. The **GMRFLib** library
   - This is a C library for fast and exact simulation of GMRFs, used to perform
     - Unconditional simulation of a GMRF;
     - Various types of conditional simulation from a GMRF;
     - Evaluation of the corresponding log-density;
     - Generation of blockupdates in MCMC-algorithms using GMRF-approximations or auxilliary variables, construction of non-Gaussian approximations to hidden GMRFs, approximate inference using INLA.
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   - A standalone C program that
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**NB:** Because the package R-INLA relies on a standalone C program, it is not available directly from CRAN
The INLA package for R — Installation

- Visit the website
  www.r-inla.org
  and follow the instructions
- The website contains source code, examples, papers and reports discussing the theory and applications of INLA
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- Visit the website [www.r-inla.org](http://www.r-inla.org) and follow the instructions.
- The website contains source code, examples, papers and reports discussing the theory and applications of INLA.
- From R, installation is performed typing
  ```r
  source("http://www.math.ntnu.no/inla/givemeINLA.R")
  ```
- Later, you can upgrade the package by typing
  ```r
  inla.upgrade()
  ```
- A test-version (which may contain unstable updates/new functions) can be obtained by typing
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  inla.upgrade(testing=TRUE)
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- R-INLA runs natively under Linux, Windows and Mac and it is possible to do multi-threading using OpenMP
The INLA package for R — How does it work?

**Input**
- Data frame, formula

**INLA package**
- Runs the `inla` program

**Produces:**
- Input files
- `.ini` files

**Collect results**

**Output**
- A R object in the class `inla`
There has been a great effort lately in producing quite a lot user-friendly(-ish) documentation.

Tutorials are (or will shortly be) available on:

- Basic INLA (probably later this year)
- SPDE (spatial models based on stochastic partial differential equations) models
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• Much of the recent development in R-INLA is devoted to extending the applications of INLA for spatial and spatio-temporal models as well as producing detailed information

• The website also has a discussion forum and a FAQ page
1. The first thing to do is to specify the model

- For example, assume we have a generic model

\[ y_i \sim p(y_i | \theta_i) \]

\[ \eta_i = g(\theta_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + f(z_i) \]

where

- \( x = (x_1, x_2) \) are observed covariates for which we are assuming a linear effect on some function \( g(\cdot) \) of the parameter \( \theta_i \)
- \( \beta = (\beta_0, \beta_1, \beta_2) \sim \text{Normal}(0, \tau_1^{-1}) \) are unstructured (“fixed”) effects
- \( z \) is an index. This can be used to include structured (“random”), spatial, spatio-temporal effect, etc.
- \( f \sim \text{Normal}(0, Q_f^{-1}(\tau_2)) \) is a suitable function used to model the structured effects
Step by step guide to using R-INLA

1. The first thing to do is to specify the model
   - For example, assume we have a generic model

\[
y_i \overset{iid}{\sim} p(y_i | \theta_i) \\
\eta_i = g(\theta_i) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + f(z_i)
\]

where
- \( x = (x_1, x_2) \) are observed covariates for which we are assuming a linear effect on some function \( g(\cdot) \) of the parameter \( \theta_i \)
- \( \beta = (\beta_0, \beta_1, \beta_2) \sim \text{Normal}(0, \tau_1^{-1}) \) are unstructured (“fixed”) effects
- \( z \) is an index. This can be used to include structured (“random”), spatial, spatio-temporal effect, etc.
- \( f \sim \text{Normal}(0, Q_f^{-1}(\tau_2)) \) is a suitable function used to model the structured effects

- As mentioned earlier, this formulation can actually be used to represent quite a wide class of models!
Step by step guide to using R-INLA

- The model is translated in R code using a `formula`
- This is sort of standard in R (you would do pretty much the same for calls to functions such as `lm`, or `glm`, or `lmer`)

\[
\text{formula} = y \sim x_1 + x_2 + f(z, \text{model}=\ldots)
\]
Step by step guide to using R-INLA

- The model is translated in R code using a **formula**
- This is sort of standard in R (you would do pretty much the same for calls to functions such as `lm`, or `glm`, or `lmer`)

\[
\text{formula} = y \sim x_1 + x_2 + f(z, \text{model}=\ldots)
\]

- The `f()` function can account for several structured effects
- This is done by specifying a different **model**
  - `iid`, `iid1d`, `iid2d`, `iid3d` specify random effects
  - `rw1`, `rw2`, `ar1` are smooth effect of covariates or time effects
  - `seasonal` specifies a seasonal effect
  - `besag` models spatially structured effects (CAR)
  - `generic` is a user-defined precision matrix
2. Call the function `inla`, specifying the data and options (more on this later), eg

```r
m = inla(formula, data=data.frame(y,x1,x2,z))
```
2. Call the function `inla`, specifying the data and options (more on this later), eg

\[
m = \text{inla(formula, data=data.frame(y,x1,x2,z))}
\]

- The data need to be included in a suitable `data.frame`
- R returns an object `m` in the class `inla`, which has some methods available
  - `summary()`
  - `plot()`
- The options let you specify the priors and hyperpriors, together with additional output
names(m)
[1] "names.fixed"  "summary.fixed"
[3] "marginals.fixed" "summary.lincomb"
[5] "marginals.lincomb" "size.lincomb"
[7] "summary.lincomb.derived" "mlik"
[9] "size.lincomb.derived" "model.random"
[11] "cpo"          "marginals.random"
[13] "summary.random" "summary.linear.predictor"
[15] "size.random"  "summary.fitted.values"
[17] "marginals.linear.predictor" "size.linear.predictor"
[19] "marginals.fitted.values" "marginals.hyperpar"
[21] "summary.hyperpar" "internal.marginals.hyperpar"
[23] "internal.summary.hyperpar" "offset.linear.predictor"
[25] "si"            "summary.spde2.blc"
[27] "model.spde2.blc" "size.spde2.blc"
[29] "marginals.spde2.blc" "misc"
[31] "logfile"      "mode"
[33] "dic"          "joint.hyper"
[35] "neffp"        "version"
[37] "nhyper"       "graph"
[39] "Q"            ".args"
[41] "cpu.used"     "model.matrix"
First, generate some data from an assumed model

\[ y_i \sim \text{Binomial}(\pi_i, N_i), \quad \text{for } i = 1, \ldots, n = 12 \]

```r
library(INLA)

# Data generation
n=12
Ntrials = sample(c(80:100), size=n, replace=TRUE)
eta = rnorm(n,0,0.5)
prob = exp(eta)/(1 + exp(eta))
y = rbinom(n, size=Ntrials, prob = prob)
data=data.frame(y=y,z=1:n,Ntrials)
```
Example — Binary data with individual random effect

data

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>37</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>36</td>
<td>3</td>
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<tr>
<td>4</td>
<td>47</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>39</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>67</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>60</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>57</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>34</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>57</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>46</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>48</td>
<td>12</td>
</tr>
</tbody>
</table>
We want to fit the following model:

\[ y_i \sim \text{Binomial}(\pi_i, N_i), \quad \text{for } i = 1, \ldots, n = 12 \]

\[ \logit(\pi_i) = \alpha + f(z_i) \]

\[ \alpha \sim \text{Normal}(0, 1000) \quad ("\text{fixed" effect}) \]

\[ f(z_i) \sim \text{Normal}(0, \sigma^2) \quad ("\text{random" effect}) \]

\[ p(\sigma^2) \propto \sigma^{-2} = \tau \quad ("\text{non-informative" prior}) \]

\[ \approx \log \sigma \sim \text{Uniform}(0, \infty) \]

Data:

<table>
<thead>
<tr>
<th>y</th>
<th>z</th>
<th>Ntrials</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50</td>
<td>1</td>
</tr>
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\[
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\[
\approx \log \sigma \sim \text{Uniform}(0, \infty)
\]

This can be done by typing in R

```r
formula = y ~ f(z,model="iid",
    hyper=list(list(prior="flat")))

m=inla(formula, data=data,
    family="binomial",
    Ntrials=Ntrials,
    control.predictor = list(compute = TRUE))

summary(m)
```
We want to fit the following model

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    family="binomial",
    Ntrials=Ntrials,
    control.predictor = list(compute = TRUE))

summary(m)
```
Example — Binary data with individual random effect

Call:
c("inla(formula = formula, family = "binomial", data = data, Ntrials = Ntrials, 
    "control.predictor = list(compute = TRUE))")

Time used:
  Pre-processing Running inla Post-processing Total
  0.2258       0.0263   0.0744    0.3264

Fixed effects:
  mean    sd  0.025quant  0.5quant  0.975quant     kld
(Intercept) -0.0021 0.136   -0.272    -0.0021     0.268 0

Random effects:
Name  Model
  z   IID model

Model hyperparameters:
  mean    sd   0.025quant  0.5quant  0.975quant
Precision for z  7.130  4.087   2.168     6.186  17.599

Expected number of effective parameters (std dev): 9.494(0.7925)
Number of equivalent replicates: 1.264

Marginal Likelihood: -54.28
CPO and PIT are computed

Posterior marginals for linear predictor and fitted values computed
Exploring the R-INLA output

Fixed effects:

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<tr>
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<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
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</thead>
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<td>0.136</td>
<td>-0.272</td>
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<td>0.268</td>
<td>0</td>
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</tbody>
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Exploring the R-INLA output

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</tbody>
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- For each unstructured ("fixed") effect, R-INLA reports a set of summary statistics from the posterior distribution.
Exploring the R-INLA output

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<td>-0.0021</td>
<td>0.268</td>
<td>0</td>
</tr>
</tbody>
</table>

- For each unstructured (“fixed”) effect, R-INLA reports a set of summary statistics from the posterior distribution.

- The value of the Kullback-Leibler divergence (KLD) describes the difference between the standard Gaussian and the Simplified Laplace Approximation to the marginal posterior densities.
  - Small values indicate that the posterior distribution is well approximated by a Normal distribution.
  - If so, the more sophisticated SLA gives a “good” error rate and therefore there is no need to use the more computationally intensive “full” Laplace approximation.
Random effects:
Name    Model
  z    IID model

Model hyperparameters:
   mean    sd    0.025quant    0.5quant    0.975quant
Precision for z   7.130    4.087    2.168    6.186    17.599

- Also for each hyper-parameter, the summary statistics are reported to describe the posterior distribution
- **NB:** INLA reports results on the precision scale (more on this later)
Expected number of effective parameters (std dev): 9.494(0.7925)
Number of equivalent replicates: 1.264
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Number of equivalent replicates: 1.264

- The expected number of effective parameters is basically the number of independent parameters included in the model
  - In a hierarchical model, because of shrinkage, information is shared across parameters
  - Example: in this case there are 14 actual parameters ($\alpha, \sigma^2, f(1), \ldots, f(12)$). However, because the structured effects are exchangeable (i.e., correlated) the “effective” number of parameters is (on average) just 9.5
Exploring the R-INLA output

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- The number of equivalent replicates indicates the available information (in terms of sample size) per effective parameter
  - Example: there are 12 data points and on average 9.5 parameters; so each is estimated using on average $12/9.5 \approx 1.3$ data points
  - Low values (with respect to the overall sample size) are indicative of poor fit
Exploring the R-INLA output

Marginal Likelihood:  -54.28
CPO and PIT are computed

- R-INLA can produce two types of “leave-one-out” measures of fit
  1. Conditional Predictive Ordinate (CPO): $p(y_i \mid y_{-i})$
     - “Extreme” values for CPO indicate a surprising observation
  2. Probability Integral Transforms (PIT): $\Pr(y_{i}^{\text{new}} \leq y_i \mid y_{-i})$
     - “Extreme” values for PIT indicate outliers
     - A histogram of PIT that does not look Uniformly distributed indicate lack of fit for the current model
Exploring the R-INLA output

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    - “Extreme” values for PIT indicate outliers
    - A histogram of PIT that does not look Uniformly distributed indicate lack of fit for the current model

- If the option
  \begin{verbatim}
  control.compute=list(cpo=TRUE)
  \end{verbatim}

is added to the call to the function \texttt{inla} then the resulting object contains values for CPO and PIT, which can then be post-processed

- \textbf{NB}: for the sake of model checking, it is useful to to increase the accuracy of the estimation for the tails of the marginal distributions
  - This can be done by adding the option
    \begin{verbatim}
    control.inla = list(strategy = "laplace", npoints = 21)
    \end{verbatim}
    to add more evaluation points (npoints=21) instead of the default npoints=9
Exploring the R-INLA output

The PIT−values, n.fail0

The CPO−values, n.fail0

Histogram of the CPO−values, n.fail0
Example — Binary data with individual random effect

```R
plot(m)

plot(m,
    plot.fixed.effects = TRUE,
    plot.lincomb = FALSE,
    plot.random.effects = FALSE,
    plot.hyperparameters = FALSE,
    plot.predictor = FALSE,
    plot.q = FALSE,
    plot.cpo = FALSE
  )

plot(m, single = TRUE)

Mean = -0.002 SD = 0.136
```

PostDens [(Intercept)]
Example — Binary data with individual random effect

plot(m)

plot(m,  
    plot.fixed.effects = FALSE, 
    plot.lincomb = FALSE, 
    plot.random.effects = TRUE, 
    plot.hyperparameters = FALSE, 
    plot.predictor = FALSE, 
    plot.q = FALSE, 
    plot.cpo = FALSE  
)
**Manipulating the results from R-INLA**

- The elements of the object \( m \) can be used for post-processing

\[
m$summary.fixed
\]

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-0.002092578</td>
<td>0.1360447</td>
<td>-0.2720331</td>
<td>-0.002101465</td>
<td>0.2680023</td>
<td>1.866805e-08</td>
</tr>
</tbody>
</table>

\[
m$summary.random
\]

\[
$z
\]

<table>
<thead>
<tr>
<th>ID</th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.117716597</td>
<td>0.2130482</td>
<td>-0.29854459</td>
<td>0.116540837</td>
<td>0.54071007</td>
<td>1.561929e-06</td>
</tr>
<tr>
<td>2</td>
<td>-0.582142549</td>
<td>0.2328381</td>
<td>-1.05855344</td>
<td>-0.575397613</td>
<td>-0.14298960</td>
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</tr>
<tr>
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<td>0.2159667</td>
<td>-0.82665552</td>
<td>-0.386498698</td>
<td>0.02359256</td>
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<td>-0.087199172</td>
<td>0.2174477</td>
<td>-0.51798771</td>
<td>-0.086259113</td>
<td>0.33838724</td>
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</tr>
<tr>
<td>5</td>
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<td>-0.03217954</td>
<td>0.388462164</td>
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</tr>
<tr>
<td>7</td>
<td>-0.145238917</td>
<td>0.2122322</td>
<td>-0.56726042</td>
<td>-0.143798605</td>
<td>0.26859415</td>
<td>2.047815e-06</td>
</tr>
<tr>
<td>8</td>
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</tr>
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<td>-0.212274011</td>
<td>0.20094086</td>
<td>4.577080e-06</td>
</tr>
<tr>
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<td>0.001622300</td>
<td>0.42152562</td>
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<tr>
<td>11</td>
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<td>-0.42961274</td>
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<td>0.43309253</td>
<td>3.843622e-09</td>
</tr>
<tr>
<td>12</td>
<td>0.580008923</td>
<td>0.2267330</td>
<td>0.15173745</td>
<td>0.573769187</td>
<td>1.04330359</td>
<td>3.191737e-05</td>
</tr>
</tbody>
</table>
Manipulating the results from R-INLA

```r
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha), t="l")
```

Marginal posterior: $p(\alpha | y)$
Manipulating the results from R-INLA

```r
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha), t="l")

inla.qmarginal(0.05, alpha)
[1] -0.2257259
```

Marginal posterior: $p(\alpha \mid y)$
`alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha), t="l")`

`inla.qmarginal(0.05, alpha)
[1] -0.2257259`

`inla.pmarginal(-.2257259, alpha)
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alpha <- m$marginals.fixed[[1]]
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[1] -0.2257259

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inla.dmarginal(0, alpha)
[1] 3.055793
```

Marginal posterior: $p(\alpha | y)$
alpha <- m$marginals.fixed[[1]]
plot(inla.smarginal(alpha),t="l")

inla.qmarginal(0.05,alpha)
[1] -0.2257259

inla.pmarginal(-0.2257259,alpha)
[1] 0.04999996

inla.dmarginal(0,alpha)
[1] 3.055793

inla.rmarginal(4,alpha)
[1] 0.05307452 0.07866796 -0.09931744 -0.02027463
**Example — Binary data with individual random effect**

**NB**: INLA works by default with **precisions**

```r
plot(m, 
    plot.fixed.effects = FALSE, 
    plot.lincomb = FALSE, 
    plot.random.effects = FALSE, 
    plot.hyperparameters = TRUE, 
    plot.predictor = FALSE, 
    plot.q = FALSE, 
    plot.cpo = FALSE 
  )
```

![PostDens [Precision for z]](image)
**Example** — Binary data with individual random effect

**NB:** INLA works by default with *precisions*

```r
plot(m,
    plot.fixed.effects = FALSE,
    plot.lincomb = FALSE,
    plot.random.effects = FALSE,
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    plot.predictor = FALSE,
    plot.q = FALSE,
    plot.cpo = FALSE
  )
```

**Problem:** usually, we want to make inference on more interpretable parameters, eg *standard deviations*
Example — Binary data with individual random effect

• Using some built-in INLA functions
  - `model$marginals.hyperpar`
  - `inla.expectation`
  - `inla.rmarginl`

it is possible to compute the structured variability, for example on the standard deviation scale, based on `nsamples` (default=1000) MC simulations from the estimated precision

```r
s <- inla.contrib.sd(m, nsamples=1000)
s$hyper

       mean      sd     2.5%      97.5%
sd for z 0.416862 0.1098968 0.2332496 0.6478648
```
Example — Binary data with individual random effect

- Using some built-in INLA functions
  - `model$marginals.hyperpar`
  - `inla.expectation`
  - `inla.rmarginal`

  It is possible to compute the structured variability, for example on the standard deviation scale, based on `nsamples` (default=1000) MC simulations from the estimated precision.

  ```r
  s <- inla.contrib.sd(m, nsamples=1000)
  s$hyper
  
  mean    sd    2.5%    97.5%
  sd for z 0.416862 0.1098968 0.2332496 0.6478648
  
  The object `s` contains a vector of simulations from the induced posterior distribution for the standard deviation scale, than can then be used for plots
  ```r
  hist(s$samples)
  plot(density(s$samples, bw=.1), xlab="sigma", main="")
  ```
Example — Binary data with individual random effect

Posterior distribution for $\sigma = \tau^{-\frac{1}{2}}$

Density

Standard deviation for the structured effect, $\sigma$
If we wanted to perform MCMC on this model, we could

1. Program it in JAGS/BUGS and save it as `model.txt`

```r
model {
  for (i in 1:n) {
    y[i] ~ dbinom(pi[i], Ntrials[i])
    logit(pi[i]) <- alpha + f[i]
    f[i] ~ dnorm(0, tau)
  }
  alpha ~ dnorm(0, .001)
  log.sigma ~ dunif(0, 10000)
  sigma <- exp(log.sigma)
  tau <- pow(sigma, -2)
}
```
If we wanted to perform MCMC on this model, we could

1. Program it in JAGS/BUGS and save it as `model.txt`

```r
model {
  for (i in 1:n) {
    y[i] ~ dbinom(pi[i], Ntrials[i])
    logit(pi[i]) <- alpha + f[i]
    f[i] ~ dnorm(0, tau)
  }
  alpha ~ dnorm(0, .001)
  log.sigma ~ dunif(0, 10000)
  sigma <- exp(log.sigma)
  tau <- pow(sigma, -2)
}
```

2. In R, use the library `R2jags` (or `R2WinBUGS`) to interface with the MCMC software

```r
library(R2jags)
filein <- "model.txt"
dataJags <- list(y=y, n=n, Ntrials=Ntrials, prec=prec)
params <- c("sigma", "tau", "f", "pi", "alpha")
inits <- function(){
  list(log.sigma=runif(1), alpha=rnorm(1), f=rnorm(n, 0, 1))
}
n.iter <- 100000
n.burnin <- 9500
n.thin <- floor((n.iter-n.burnin)/500)
mj <- jags(dataJags, inits, params, model.file=filein, n.chains=2, n.iter, n.burnin, n.thin, DIC=TRUE, working.directory=working.dir, progress.bar="text")
print(mj, digits=3, intervals=c(0.025, 0.975))
```
Example — Binary data with individual random effect

Inference for Bugs model at "model.txt", fit using jags, 
2 chains, each with 1e+05 iterations (first 9500 discarded), n.thin = 181 
n.sims = 1000 iterations saved  
(Time to run: 4.918 sec)

<table>
<thead>
<tr>
<th></th>
<th>mu.vect</th>
<th>sd.vect</th>
<th>2.5%</th>
<th>97.5%</th>
<th>Rhat</th>
<th>n.eff</th>
</tr>
</thead>
<tbody>
<tr>
<td>alpha</td>
<td>-0.005</td>
<td>0.146</td>
<td>-0.270</td>
<td>0.292</td>
<td>1.001</td>
<td>1000</td>
</tr>
<tr>
<td>f[1]</td>
<td>0.122</td>
<td>0.220</td>
<td>-0.347</td>
<td>0.582</td>
<td>1.001</td>
<td>1000</td>
</tr>
<tr>
<td>f[2]</td>
<td>-0.564</td>
<td>0.238</td>
<td>-1.051</td>
<td>-0.115</td>
<td>1.008</td>
<td>190</td>
</tr>
<tr>
<td>f[3]</td>
<td>-0.386</td>
<td>0.229</td>
<td>-0.880</td>
<td>0.050</td>
<td>1.000</td>
<td>1000</td>
</tr>
<tr>
<td>f[4]</td>
<td>-0.086</td>
<td>0.225</td>
<td>-0.549</td>
<td>0.367</td>
<td>1.002</td>
<td>780</td>
</tr>
<tr>
<td>f[5]</td>
<td>0.392</td>
<td>0.227</td>
<td>-0.047</td>
<td>0.828</td>
<td>1.002</td>
<td>870</td>
</tr>
<tr>
<td>f[6]</td>
<td>-0.351</td>
<td>0.229</td>
<td>-0.805</td>
<td>0.081</td>
<td>1.000</td>
<td>1000</td>
</tr>
<tr>
<td>f[7]</td>
<td>-0.141</td>
<td>0.221</td>
<td>-0.578</td>
<td>0.286</td>
<td>1.001</td>
<td>1000</td>
</tr>
<tr>
<td>f[8]</td>
<td>0.672</td>
<td>0.236</td>
<td>0.246</td>
<td>1.200</td>
<td>1.002</td>
<td>860</td>
</tr>
<tr>
<td>f[9]</td>
<td>-0.224</td>
<td>0.210</td>
<td>-0.643</td>
<td>0.178</td>
<td>1.000</td>
<td>1000</td>
</tr>
<tr>
<td>f[10]</td>
<td>0.016</td>
<td>0.219</td>
<td>-0.396</td>
<td>0.463</td>
<td>1.006</td>
<td>1000</td>
</tr>
<tr>
<td>f[11]</td>
<td>-0.001</td>
<td>0.221</td>
<td>-0.441</td>
<td>0.416</td>
<td>1.002</td>
<td>780</td>
</tr>
<tr>
<td>f[12]</td>
<td>0.585</td>
<td>0.245</td>
<td>0.153</td>
<td>1.093</td>
<td>1.001</td>
<td>1000</td>
</tr>
<tr>
<td>sigma</td>
<td>0.414</td>
<td>0.120</td>
<td>0.230</td>
<td>0.693</td>
<td>1.000</td>
<td>1000</td>
</tr>
<tr>
<td>tau</td>
<td>7.415</td>
<td>4.546</td>
<td>2.080</td>
<td>18.951</td>
<td>1.000</td>
<td>1000</td>
</tr>
<tr>
<td>deviance</td>
<td>72.378</td>
<td>5.497</td>
<td>64.016</td>
<td>84.715</td>
<td>1.000</td>
<td>1000</td>
</tr>
</tbody>
</table>

For each parameter, n.eff is a crude measure of effective sample size, 
and Rhat is the potential scale reduction factor (at convergence, Rhat=1).

DIC info (using the rule, pD = var(deviance)/2) 
pD = 15.1 and DIC = 87.5 
DIC is an estimate of expected predictive error (lower deviance is better).
Example — Binary data with individual random effect

Structured effects

-1.0 −0.5 0.0 0.5 1.0

f(1)
f(2)
f(3)
f(4)
f(5)
f(6)
f(7)
f(8)
f(9)
f(10)
f(11)
f(12)

JAGS
INLA
Example — Binary data with individual random effect

Posterior distribution for $\sigma = \tau^{-\frac{1}{2}}$

![Graph showing the posterior distribution for $\sigma = \tau^{-\frac{1}{2}}$]
• **R-INLA** allows to make predictive inference based on the observed model
• Suppose for example that the \((n + 1)\)-th value is not (yet) observed for the response variable \(y\)
  – **NB**: for **R-INLA**, a missing value in the response means no likelihood contribution
Example — Binary data with individual random effect

- R-INLA allows to make predictive inference based on the observed model
- Suppose for example that the \((n + 1)\)-th value is not (yet) observed for the response variable \(y\)
  - **NB**: for R-INLA, a missing value in the response means no likelihood contribution

- We can code this in R, by augmenting the original dataset

```r
y[n+1] <- NA
Ntrials[n+1] <- sample(c(80:100),size=1,replace=TRUE)
data2 <- data.frame(y=y,z=1:(n+1),Ntrials=Ntrials)

formula2 = y ~ f(z,model="iid",hyper=list(list(prior="flat")))
m2=inla(formula2,data=data2,
    family="binomial",
    Ntrials=Ntrials,
    control.predictor = list(compute = TRUE))

summary(m2)
```
Example — Binary data with individual random effect

Time used:

<table>
<thead>
<tr>
<th></th>
<th>Pre-processing</th>
<th>Running inla</th>
<th>Post-processing</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0883</td>
<td>0.0285</td>
<td>0.0236</td>
<td>0.1404</td>
</tr>
<tr>
<td></td>
<td>(0.2258)</td>
<td>(0.0263)</td>
<td>(0.0744)</td>
<td>(0.3264)</td>
</tr>
</tbody>
</table>

Fixed effects:

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-0.0021</td>
<td>0.136</td>
<td>-0.272</td>
<td>-0.0021</td>
<td>0.268</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(-0.0021)</td>
<td>(0.136)</td>
<td>(-0.272)</td>
<td>(-0.0021)</td>
<td>(0.268)</td>
<td>(0)</td>
</tr>
</tbody>
</table>

Random effects:

<table>
<thead>
<tr>
<th>Name</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>z</td>
<td>IID model</td>
</tr>
</tbody>
</table>

Model hyperparameters:

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precision for z</td>
<td>7.130</td>
<td>4.087</td>
<td>2.168</td>
<td>6.186</td>
<td>17.599</td>
</tr>
<tr>
<td></td>
<td>(7.130)</td>
<td>(4.087)</td>
<td>(2.168)</td>
<td>(6.186)</td>
<td>(17.599)</td>
</tr>
</tbody>
</table>

Expected number of effective parameters(std dev): 9.494(0.7925)
Number of equivalent replicates : 1.264

Marginal Likelihood:  -54.28
CPO and PIT are computed

Posterior marginals for linear predictor and fitted values computed
The estimated value for the predictive distribution can be retrieved using the following code:

```r
pred <- m2$marginals.linear.predictor[[n+1]]
plot(pred, xlab="", ylab="Density")
lines(inla.smarginal(pred))
```

which can be used to generate, eg a graph of the predictive density.
Specifying the model — options

- It is possible to specify link functions that are different from the default used by R-INLA
- This is done by specifying suitable values for the option \texttt{control.family} to the call to \texttt{inla}, eg

```r
m = inla(formula, data=data, family="binomial", Ntrials=Ntrials,
          control.predictor=list(compute=TRUE),
          control.family = list(link = "probit"))
```
• It is possible to specify link functions that are different from the default used by R-INLA
• This is done by specifying suitable values for the option `control.family` to the call to `inla`, eg

\[
m = \text{inla}(\text{formula, data=data, family=\text{"binomial"}}, \text{Ntrials=Ntrials, control.predictor=list(compute=TRUE), control.family = list(link = \text{"probit\")}}
\]

• More details are available on the R-INLA website:
  - http://www.r-inla.org/models/likelihoods
Specifying the model — options

- **R-INLA** has a set of default priors for the different components of the LGM/GMRF.
- For example, in a standard hierarchical formulation, **R-INLA** assumes:
  - Unstructured ("fixed") effects: \( \beta \sim \text{Normal}(0, 0.001) \)
  - Structured ("random") effects: \( f(z_i) \sim \text{Normal}(0, \tau) \)
    \[
    \log \tau \sim \text{logGamma}(1, 0.00005)
    \]
- **NB**: It is possible to see the default settings using the function

  \[
  \text{inla.model.properties(<name>, <section>)}
  \]
Specifying the model — options

- **R-INLA** has a set of default priors for the different components of the LGM/GMRF
- For example, in a standard hierarchical formulation, **R-INLA** assumes
  - Unstructured ("fixed") effects: $\beta \sim \text{Normal}(0, 0.001)$
  - Structured ("random") effects: $f(z_i) \sim \text{Normal}(0, \tau)$
    $$\log \tau \sim \text{logGamma}(1, 0.00005)$$
- **NB**: It is possible to see the default settings using the function
  
  ```r
  inla.model.properties(<name>, <section>)
  ```

- However, there is a wealth of possible formulations that the user can specify, especially for the hyperpriors
- More details are available on the **R-INLA** website:
  - [http://www.r-inla.org/models/likelihoods](http://www.r-inla.org/models/likelihoods)
  - [http://www.r-inla.org/models/latent-models](http://www.r-inla.org/models/latent-models)
  - [http://www.r-inla.org/models/priors](http://www.r-inla.org/models/priors)
### Models for the observed data

<table>
<thead>
<tr>
<th>Model</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Negative Binomial</td>
<td>nbinomial</td>
</tr>
<tr>
<td>Poisson</td>
<td>poisson</td>
</tr>
<tr>
<td>Binomial</td>
<td>binomial</td>
</tr>
<tr>
<td>Clustered Binomial</td>
<td>cbinomial</td>
</tr>
<tr>
<td>Gaussian</td>
<td>gaussian</td>
</tr>
<tr>
<td>Skew Normal</td>
<td>sn</td>
</tr>
<tr>
<td>Laplace</td>
<td>laplace</td>
</tr>
<tr>
<td>Student-t</td>
<td>T</td>
</tr>
<tr>
<td>Gaussian model for stochastic volatility</td>
<td>stochvol</td>
</tr>
<tr>
<td>Student-t model for stochastic volatility</td>
<td>stochvol.t</td>
</tr>
<tr>
<td>NIG model for stochastic volatility</td>
<td>stochvol.nig</td>
</tr>
<tr>
<td>Zero inflated Poisson</td>
<td>zeroinflated.poisson.x(x=0,1,2)</td>
</tr>
<tr>
<td>Zero inflated Binomial</td>
<td>zeroinflated.binomial.x(x=0,1)</td>
</tr>
<tr>
<td>Zero inflated negative Binomial</td>
<td>zeroinflated.nbinomial.x(x=0,1,2)</td>
</tr>
<tr>
<td>Zero inflated beta binomial (type 2)</td>
<td>zeroinflated.betabinomial.2</td>
</tr>
<tr>
<td>Generalised extreme value distribution (GEV)</td>
<td>gev</td>
</tr>
<tr>
<td>Beta</td>
<td>beta</td>
</tr>
<tr>
<td>Gamma</td>
<td>gamma</td>
</tr>
<tr>
<td>Beta-Binomial</td>
<td>betabinomial</td>
</tr>
<tr>
<td>Logistic distribution</td>
<td>logistic</td>
</tr>
<tr>
<td>Exponential (Survival models)</td>
<td>exponential</td>
</tr>
<tr>
<td>Weibull (Survival model)</td>
<td>weibull</td>
</tr>
<tr>
<td>LogLogistic (Survival model)</td>
<td>loglogistic</td>
</tr>
<tr>
<td>LogNormal (Survival model)</td>
<td>lognormal</td>
</tr>
<tr>
<td>Cox model (Survival model)</td>
<td>coxph</td>
</tr>
</tbody>
</table>
## Specifying the model — options

### Models for the GMRF

<table>
<thead>
<tr>
<th>Model</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Independent random variables</td>
<td>iid</td>
</tr>
<tr>
<td>Linear</td>
<td>linear</td>
</tr>
<tr>
<td>Random walk of order 1</td>
<td>rw1</td>
</tr>
<tr>
<td>Random walk of order 2</td>
<td>rw2</td>
</tr>
<tr>
<td>Continuous random walk of order 2</td>
<td>crw2</td>
</tr>
<tr>
<td>Model for seasonal variation</td>
<td>seasonal</td>
</tr>
<tr>
<td>Model for spatial effect</td>
<td>besag</td>
</tr>
<tr>
<td>Model for spatial effect</td>
<td>besagproper</td>
</tr>
<tr>
<td>Model for weighted spatial effects</td>
<td>besag2</td>
</tr>
<tr>
<td>Model for spatial effect + random effect</td>
<td>bym</td>
</tr>
<tr>
<td>Autoregressive model of order 1</td>
<td>ar1</td>
</tr>
<tr>
<td>Autoregressive model of order p</td>
<td>ar</td>
</tr>
<tr>
<td>The Ornstein-Uhlenbeck process</td>
<td>ou</td>
</tr>
<tr>
<td>User defined structure matrix, type 0</td>
<td>generic0</td>
</tr>
<tr>
<td>User defined structure matrix, type1</td>
<td>generic1</td>
</tr>
<tr>
<td>User defined structure matrix, type2</td>
<td>generic2</td>
</tr>
<tr>
<td>Model for correlated effects with Wishart prior (dimension 1, 2, 3, 4 and 5)</td>
<td>iid1d, iid2d, iid3d, iid4d, iid5d</td>
</tr>
<tr>
<td>(Quite) general latent model</td>
<td>z</td>
</tr>
<tr>
<td>Random walk of 2nd order on a lattice</td>
<td>rw2d</td>
</tr>
<tr>
<td>Gaussian field with Matern covariance function</td>
<td>matern2d</td>
</tr>
<tr>
<td>Classical measurement error model</td>
<td>mec</td>
</tr>
<tr>
<td>Berkson measurement error model</td>
<td>meb</td>
</tr>
</tbody>
</table>
Models for the hyper-parameters

<table>
<thead>
<tr>
<th>Model</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal distribution</td>
<td>normal, gaussian</td>
</tr>
<tr>
<td>Log-gamma distribution</td>
<td>loggamma</td>
</tr>
<tr>
<td>Improper flat prior</td>
<td>flat</td>
</tr>
<tr>
<td>Truncated Normal distribution</td>
<td>logtnormal, logtgaussian</td>
</tr>
<tr>
<td>Improper flat prior on the log scale</td>
<td>logflat</td>
</tr>
<tr>
<td>Improper flat prior on the 1/ log scale</td>
<td>logiflat</td>
</tr>
<tr>
<td>Wishart prior</td>
<td>wishart</td>
</tr>
<tr>
<td>Beta for correlations</td>
<td>betacorrelation</td>
</tr>
<tr>
<td>Logit of a Beta</td>
<td>logitbeta</td>
</tr>
<tr>
<td>Define your own prior</td>
<td>expression:</td>
</tr>
</tbody>
</table>
Hyper-parameters (eg correlation coefficients $\rho$ or precisions $\tau$) are represented internally using a suitable transformation, eg

$$\psi_1 = \log(\tau)$$

or

$$\psi_2 = \log\left(\frac{1 + \rho}{1 - \rho}\right)$$

to improve symmetry and approximate Normality
Hyper-parameters (e.g., correlation coefficients $\rho$ or precisions $\tau$) are represented internally using a suitable transformation, e.g.

$$\psi_1 = \log(\tau)$$

or

$$\psi_2 = \log\left(\frac{1 + \rho}{1 - \rho}\right)$$

to improve symmetry and approximate Normality.

- Initial values are given on the internal scale.
- Priors are also defined on the internal scale.
- So, when specifying custom values, care is needed!
Specifying the prior (1)

Consider the model

\[
y_i \mid \theta_i, \sigma^2 \sim \text{Normal}(\theta_i, \sigma^2)
\]

\[
\theta_i = \alpha + \beta x_i
\]

\[
\alpha, \beta \overset{\text{iid}}{\sim} \text{Normal}(0, 0.001)
\]

\[
\log \tau = -\log \sigma^2 \sim \text{logGamma}(1, 0.01)
\]

n=100
a = 1; b = 1
x = rnorm(n)
eta = a + b*x
tau = 100
scale = exp(rnorm(n))
prec = scale*tau
y = rnorm(n, mean = eta, sd = 1/sqrt(prec))
data = list(y=y, x=x)
formula = y ~ 1 + x
result = inla(formula, family = "gaussian", data = data,
control.family = list(hyper = list(
    prec = list(prior = "loggamma",param = c(1,0.01),initial = 2))),
scale=scale, keep=TRUE)
summary(result)
Specifying the prior (1)

Time used:

- Pre-processing: 0.0776
- Running `inla`: 0.0828
- Post-processing: 0.0189
- Total: 0.1793

Fixed effects:

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>1.0013</td>
<td>0.0074</td>
<td>0.9868</td>
<td>1.0013</td>
<td>1.0158</td>
<td>0</td>
</tr>
<tr>
<td>x</td>
<td>0.9936</td>
<td>0.0075</td>
<td>0.9788</td>
<td>0.9936</td>
<td>1.0083</td>
<td>0</td>
</tr>
</tbody>
</table>

The model has no random effects

Model hyperparameters:

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precision for the Gaussian observations</td>
<td>108.00</td>
<td>15.34</td>
<td>80.60</td>
<td>107.09</td>
<td>140.74</td>
</tr>
</tbody>
</table>

Expected number of effective parameters (std dev): 2.298(0.0335)
Number of equivalent replicates: 43.52

Marginal Likelihood: 83.86
CPO and PIT are computed
Consider the model

\[ y_i | \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2) \]
\[ \mu \sim \text{Normal}(0, 0.001) \]
\[ \log \tau = -\log \sigma^2 \sim \text{Normal}(0, 1) \]

```r
n = 10
y = rnorm(n)
formula = y ~ 1
result = inla(formula, data = data.frame(y),
              control.family = list(hyper = list(
                                 prec = list(prior = "normal",param = c(0,1))))
)
summary(result)
```
Consider the model

\[
\begin{align*}
y_i \mid \mu, \sigma^2 & \sim \text{Normal}(\mu, \sigma^2) \\
\mu & \sim \text{Normal}(0, 0.001) \\
\log \tau = -\log \sigma^2 & \sim \text{Normal}(0, 1)
\end{align*}
\]

\[
\begin{align*}
n &= 10 \\
y &= \text{rnorm}(n) \\
\text{formula} &= y \sim 1 \\
\text{result} &= \text{inla(formula, data = data.frame(y),} \\
&\hspace{1cm}\text{control.family = list(hyper = list(} \\
&\hspace{2cm}\text{prec = list(prior = "normal",param = c(0,1)))))} \\
\text{summary(result)}
\end{align*}
\]

- **NB**: INLA thinks in terms of LGMs and GMRFs
- Thus, the common mean for all the observations is specified in terms of a regression!
Time used:
Pre-processing Running inla Post-processing Total
0.0740 0.0214 0.0221 0.1175

Fixed effects:

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-0.3853</td>
<td>0.4077</td>
<td>-1.1939</td>
<td>-0.3853</td>
<td>0.4237</td>
<td>0</td>
</tr>
</tbody>
</table>

The model has no random effects

Model hyperparameters:

<table>
<thead>
<tr>
<th>Precision for the Gaussian observations</th>
<th>mean</th>
<th>sd</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6512</td>
<td>0.268</td>
<td>0.2590</td>
<td>0.6089</td>
<td>1.2919</td>
<td></td>
</tr>
</tbody>
</table>

Expected number of effective parameters (std dev): 1.00(0.00)
Number of equivalent replicates: 9.999

Marginal Likelihood: -17.30
CPO and PIT are computed
Running the model in JAGS

```r
model {
  for (i in 1:n) {
    y[i] ~ dnorm(mu,tau)
  }
  mu ~ dnorm(0,0.001)
  log.tau ~ dnorm(0,1)
  tau <- exp(log.tau)
}
```

produces similar results

Inference for Bugs model at "modelHyperPriorNormal.txt", fit using jags,
2 chains, each with 1e+05 iterations (first 9500 discarded), n.thin = 181
n.sims = 1000 iterations saved
  mu.vect  sd.vect  2.5%  97.5%  Rhat  n.eff
  mu     -0.384   0.447  -1.293  0.555   1.000    1000
  tau     0.642   0.258   0.233  1.240   1.006    1000
  deviance 34.507  1.930  32.645  39.897   1.002    650

For each parameter, n.eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor (at convergence, Rhat=1).

DIC info (using the rule, pD = var(deviance)/2)
pD = 1.9 and DIC = 36.4
DIC is an estimate of expected predictive error (lower deviance is better).
We can also assume different priors for the unstructured ("fixed") effects, e.g., suppose we want to fit the model

\[
y_i \mid \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2)
\]

\[
\mu \sim \text{Normal}(10, 4)
\]

\[
\log \tau = -\log \sigma^2 \sim \text{Normal}(0, 1)
\]
Specifying the prior (2)

- We can also assume different priors for the unstructured ("fixed") effects, e.g., suppose we want to fit the model

\[ y_i \mid \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2) \]
\[ \mu \sim \text{Normal}(10, 4) \]
\[ \log \tau = -\log \sigma^2 \sim \text{Normal}(0, 1) \]

- This can be done by using the option `control.fixed`, e.g.

```r
result = inla(formula, data = data.frame(y),
               control.family = list(hyper = list(
                prec = list(prior = "normal",param = c(0, 1))))
               control.fixed=list(
                mean.intercept=10,prec.intercept=4)
)
```
We can also assume different priors for the unstructured ("fixed") effects, eg suppose we want to fit the model

\[ y_i | \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2) \]
\[ \mu \sim \text{Normal}(10, 4) \]
\[ \log \tau = -\log \sigma^2 \sim \text{Normal}(0, 1) \]

This can be done by using the option `control.fixed`, eg

```r
result = inla(formula, data = data.frame(y),
              control.family = list(hyper = list(pec = list(prior = "normal",param = c(0, 1))))
              control.fixed=list(
              mean.intercept=10,prec.intercept=4)
)
```

**NB**: If the model contains fixed effects for some covariates, non-default priors can be included using the option

```r
control.fixed=list(mean=list(value),prec=list(value))
```
Specifying the prior (2)

Time used:

<table>
<thead>
<tr>
<th></th>
<th>Pre-processing</th>
<th>Running inla</th>
<th>Post-processing</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0747</td>
<td>0.0311</td>
<td>0.0164</td>
<td>0.1222</td>
</tr>
</tbody>
</table>

Fixed effects:

<table>
<thead>
<tr>
<th>Mean</th>
<th>SD</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>9.5074</td>
<td>0.502</td>
<td>8.5249</td>
<td>9.5067</td>
<td>10.4935</td>
</tr>
<tr>
<td></td>
<td>-0.3853</td>
<td>0.407</td>
<td>-1.1939</td>
<td>-0.3853</td>
<td>0.4237</td>
</tr>
</tbody>
</table>

The model has no random effects

Model hyperparameters:

<table>
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<tr>
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<th>SD</th>
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<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precision</td>
<td>0.0218</td>
<td>0.007</td>
<td>0.0105</td>
<td>0.0208</td>
<td>0.0391</td>
</tr>
<tr>
<td>(Observations)</td>
<td>0.268</td>
<td>0.2590</td>
<td>0.6089</td>
<td>1.2919</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Expected number of effective parameters(std dev): 0.0521(0.0129)

Number of equivalent replicates : 192.05

Marginal Likelihood: 153.84

CPO and PIT are computed
• As mentioned earlier, for computational reasons, by default INLA uses a relatively rough grid to estimate the marginal posterior for the hyperparameters $p(\psi | y)$
• This is generally good enough, but the procedure can be refined
• As mentioned earlier, for computational reasons, by default INLA uses a relatively rough grid to estimate the marginal posterior for the hyperparameters $p(\psi | y)$.

• This is generally good enough, but the procedure can be refined.

• After the model has been estimated using the standard procedure, it is possible to increase precision in the estimation by re-fitting it using the command:

\[
\text{inla.hyperpar}(m, \text{options})
\]

• This modifies the estimation for the hyperparameters and (potentially, but not necessarily!) that for the parameters.
A more complex model

- Consider the classic model for seizure counts in a RCT of anti-conversant therapy in epilepsy ("Epil" in the **BUGS** manual)

- The data are as follows

<table>
<thead>
<tr>
<th>Patient</th>
<th>Visit 1</th>
<th>Visit 2</th>
<th>Visit 3</th>
<th>Visit 4</th>
<th>Trt</th>
<th>Base</th>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>11</td>
<td>31</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>11</td>
<td>30</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>59</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>12</td>
<td>37</td>
</tr>
</tbody>
</table>

- We replicate the model presented in the **BUGS** manual, which uses slightly modified version of the covariates
A more complex model

We model

\[ y_{jk} \sim \text{Poisson}(\mu_{jk}) \]
\[ \log(\mu_{jk}) = \alpha_0 + \alpha_1 \log(B_j/4) + \alpha_2 \text{Trt}_j + \]
\[ \alpha_3 \text{Trt}_j \times \log(B_j/4) + \alpha_4 \log(\text{Age}_j) + \]
\[ \alpha_5 V_{4k} + u_j + v_{ik} \]
\[ \alpha_0, \ldots \alpha_5 \overset{iid}{\sim} \text{Normal}(0, \tau_{\alpha}), \quad \tau_{\alpha} \text{ known} \]
\[ u_j \sim \text{Normal}(0, \tau_u), \quad \tau_u \sim \text{Gamma}(a_u, b_u) \]
\[ v_{jk} \sim \text{Normal}(0, \tau_v), \quad \tau_v \sim \text{Gamma}(a_v, b_v) \]
A more complex model

We model

$$y_{jk} \sim \text{Poisson} (\mu_{jk})$$

$$\log(\mu_{jk}) = \alpha_0 + \alpha_1 \log(B_j/4) + \alpha_2 \text{Trt}_j +$$

$$\alpha_3 \text{Trt}_j \times \log(B_j/4) + \alpha_4 \log(\text{Age}_j) +$$

$$\alpha_5 V_{4k} + u_j + v_{jk}$$

$$\alpha_0, \ldots \alpha_5 \overset{iid}{\sim} \text{Normal}(0, \tau_\alpha), \quad \tau_\alpha \text{ known}$$

$$u_j \sim \text{Normal}(0, \tau_u), \quad \tau_u \sim \text{Gamma}(a_u, b_u)$$

$$v_{jk} \sim \text{Normal}(0, \tau_v), \quad \tau_v \sim \text{Gamma}(a_v, b_v)$$

$$\alpha = (\alpha_0, \ldots \alpha_5)$$ indicates a set of “fixed” effects for the relevant (re-scaled) covariates
A more complex model

We model

\[ y_{jk} \sim \text{Poisson}(\mu_{jk}) \]
\[ \log(\mu_{jk}) = \alpha_0 + \alpha_1 \log(B_j/4) + \alpha_2 \text{Trt}_j + \]
\[ \alpha_3 \text{Trt}_j \times \log(B_j/4) + \alpha_4 \log(Age_j) + \]
\[ \alpha_5 V_{4k} + u_j + v_{jk} \]

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\[ v_{jk} \sim \text{Normal}(0, \tau_v), \quad \tau_v \sim \text{Gamma}(a_v, b_v) \]

\[ \alpha = (\alpha_0, \ldots \alpha_5) \] indicates a set of “fixed” effects for the relevant (re-scaled) covariates

\[ u_j \] is an individual “random” effect
A more complex model

We model

\[ y_{jk} \sim \text{Poisson}(\mu_{jk}) \]
\[ \log(\mu_{jk}) = \alpha_0 + \alpha_1 \log(B_j/4) + \alpha_2 \text{Trt}_j + \alpha_3 \text{Trt}_j \times \log(B_j/4) + \alpha_4 \log(Age_j) + \alpha_5 V_{4k} + u_j + v_{ik} \]

\[ \alpha_0, \ldots, \alpha_5 \sim \text{Normal}(0, \tau_\alpha), \quad \tau_\alpha \text{ known} \]
\[ u_j \sim \text{Normal}(0, \tau_u), \quad \tau_u \sim \text{Gamma}(a_u, b_u) \]
\[ v_{jk} \sim \text{Normal}(0, \tau_v), \quad \tau_v \sim \text{Gamma}(a_v, b_v) \]

\( \alpha = (\alpha_0, \ldots, \alpha_5) \) indicates a set of “fixed” effects for the relevant (re-scaled) covariates

\( u_j \) is an individual “random” effect

\( v_{jk} \) is a subject by visit “random” effect, which accounts for extra-Poisson variability
A more complex model

```r
data(Epil)
head(Epil,n=3)
  y Trt Base Age V4 rand Ind
  1 5 0 11 31 0 1 1
  2 3 0 11 31 0 2 1
  3 3 0 11 31 0 3 1

formula <- y ~ log(Base/4) + Trt +
            I(Trt * log(Base/4)) + log(Age) + V4 +
            f(Ind, model = "iid") + f(rand, model="iid")

m <- inla(formula, family="poisson", data = Epil)
```

- **NB**: The variable `Ind` indicates the individual random effect \( u_j \), while the variable `rand` is used to model the subject by visit random effect \( v_{jk} \).

- Interactions can be indicated in the \( \mathsf{R} \) formula using the notation
  \[
  \text{I}(\text{var1} \times \text{var2})
  \]

- The model assumes that the two structured effects are independent. This can be relaxed and a joint model can be used instead.
### Pre-processing, Running inla, Post-processing, Total

<table>
<thead>
<tr>
<th></th>
<th>Pre-processing</th>
<th>Running inla</th>
<th>Post-processing</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.3672</td>
<td>0.2780</td>
<td>0.1276</td>
<td>0.7728</td>
</tr>
</tbody>
</table>

### Fixed effects:

<table>
<thead>
<tr>
<th>Effect</th>
<th>Mean</th>
<th>SD</th>
<th>0.025quant</th>
<th>0.5quant</th>
<th>0.975quant</th>
<th>kld</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-1.3877</td>
<td>1.2107</td>
<td>-3.7621</td>
<td>-1.3913</td>
<td>1.0080</td>
<td>0.0055</td>
</tr>
<tr>
<td>log(Base/4)</td>
<td>0.8795</td>
<td>0.1346</td>
<td>0.6144</td>
<td>0.8795</td>
<td>1.1447</td>
<td>0.0127</td>
</tr>
<tr>
<td>Trt</td>
<td>-0.9524</td>
<td>0.4092</td>
<td>-1.7605</td>
<td>-0.9513</td>
<td>-0.1498</td>
<td>0.0021</td>
</tr>
<tr>
<td>I(Trt * log(Base/4))</td>
<td>0.3506</td>
<td>0.2081</td>
<td>-0.0586</td>
<td>0.3504</td>
<td>0.7611</td>
<td>0.0011</td>
</tr>
<tr>
<td>log(Age)</td>
<td>0.4830</td>
<td>0.3555</td>
<td>-0.2206</td>
<td>0.4843</td>
<td>1.1798</td>
<td>0.0007</td>
</tr>
<tr>
<td>V4</td>
<td>-0.1032</td>
<td>0.0853</td>
<td>-0.2705</td>
<td>-0.1032</td>
<td>0.0646</td>
<td>0.0003</td>
</tr>
</tbody>
</table>

### Random effects:

<table>
<thead>
<tr>
<th>Name</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ind</td>
<td>IID model</td>
</tr>
<tr>
<td>rand</td>
<td>IID model</td>
</tr>
</tbody>
</table>

### Model hyperparameters:

<table>
<thead>
<tr>
<th></th>
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<th>0.975quant</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precision for Ind</td>
<td>4.635</td>
<td>1.343</td>
<td>2.591</td>
<td>4.436</td>
<td>7.808</td>
</tr>
<tr>
<td>Precision for rand</td>
<td>8.566</td>
<td>2.115</td>
<td>5.206</td>
<td>8.298</td>
<td>13.458</td>
</tr>
</tbody>
</table>

Expected number of effective parameters (std dev): 118.97(8.586)
Number of equivalent replicates: 1.984

Marginal Likelihood: -670.55
A more complex model

<table>
<thead>
<tr>
<th>A more complex model</th>
<th>Pre-processing</th>
<th>Running inla</th>
<th>Post-processing</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.3672</td>
<td>0.2780</td>
<td>0.1276</td>
<td>0.7728</td>
</tr>
</tbody>
</table>

(MCMC: approximately 30 mins)

Fixed effects:

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>sd</th>
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<tbody>
<tr>
<td>Ind</td>
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<tr>
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</tr>
</tbody>
</table>

Model hyperparameters:

<table>
<thead>
<tr>
<th></th>
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Expected number of effective parameters(std dev): 118.97(8.586)
Number of equivalent replicates : 1.984
Marginal Likelihood:  -670.55
Conclusions

• Integrated Nested Laplace Approximation is a very effective tool to estimate LGMs
  – Estimation time can be much lower than for standard MCMC
  – Precision of estimation is usually higher than for standard MCMC
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- MCMC still provides a slightly more flexible approach
  - Virtually any model can be fit using JAGS/BUGS
  - The range of priors available is wider in an MCMC setting than in INLA
  - Documentation and examples is more extensive for standard MCMC models
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  - Virtually any model can be fit using JAGS/BUGS
  - The range of priors available is wider in an MCMC setting than in INLA
  - Documentation and examples is more extensive for standard MCMC models

- Nevertheless, INLA proves to be a very flexible tool, which is able to fit a very wide range of models
  - Generalised linear (mixed) models
  - Log-Gaussian Cox processes
  - Survival analysis
  - Spline smoothing
  - Spatio-temporal models

- The INLA setup can be highly specialised (choice of data models, priors and hyperpriors) although this is a bit less intuitive than (most) MCMC models
Thank you!